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A GENERALIZATION OF THE STRICT TOPOLOGY(1)

BY
ROBIN GILES

Abstract. The strict topology β on the space $C(X)$ of bounded real-valued continuous functions on a topological space X was defined, for locally compact X, by Buck (Michigan Math. J. 5 (1958), 95–104). Among other things he showed that (a) $C(X)$ is β-complete, (b) the dual of $C(X)$ under the strict topology is the space of all finite signed regular Borel measures on X, and (c) a Stone-Weierstrass theorem holds for β-closed subalgebras of $C(X)$. In this paper the definition of the strict topology is generalized to cover the case of an arbitrary topological space and these results are established under the following conditions on X: for (a) X is a k-space; for (b) X is completely regular; for (c) X is unrestricted.

1. **Introduction and notation.** Let X be a topological space. We denote by $B(X)$ the algebra of all bounded real-valued functions on X and by $C(X)$ the subalgebra of $B(X)$ consisting of continuous functions. $B_0(X)$ denotes the ideal in $B(X)$ consisting of functions *vanishing at infinity*, in that for any $\varepsilon > 0$ there is a compact set $K \subset X$ such that $|f(x)| < \varepsilon$ for $x \notin K$, and $C_0(X) = B_0(X) \cap C(X)$. Note that $B_0(X)$ contains

(a) the characteristic function $\chi(K)$ of each compact set $K \subset X$;

(b) every function ψ of the form $\psi = \sum_{n=1}^{\infty} \alpha_n \chi(K_n)$, where $\alpha_n \geqq 0$ for all n, $\alpha_n \to 0$, and the sets K_n are compact and disjoint; and so, in particular,

(c) the function $\psi = \sum_{n=1}^{\infty} \alpha_n \chi(\{x_n\})$, where (x_n) is any sequence of distinct points.

We shall need the following lemma:

LEMMA 1.1. *If f is a real-valued function on X and $f\psi$ is bounded for every ψ in $B_0(X)$ then f is bounded.*

Proof. Suppose f is not bounded. Choose a sequence (x_n) in X with $|f(x_n)| \to \infty$ and put $\psi = \sum |f(x_n)|^{-1/2} \chi(\{x_n\})$. Then $\psi \in B_0(X)$ but $f\psi$ is not bounded.

The strict topology on $C(X)$ was defined for locally compact X by Buck [1] by means of a set of seminorms determined by the elements of $C_0(X)$. If X is completely regular but not locally compact, $C_0(X)$ may be very small (for instance, if X is the rationals [3, p. 109]) and does not yield a useful topology for $C(X)$. We claim that in this case the natural generalization of the strict topology is obtained by letting the role of $C_0(X)$ be played by $B_0(X)$; the change makes no difference if

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X is locally compact. Indeed, under mild conditions on X, we shall prove that, with this generalized strict topology, $C(X)$ is complete (Theorem 2.4) and its dual is the space of bounded signed regular Borel measures on X (Theorem 4.6), and we shall establish a Stone-Weierstrass theorem for $C(X)$ (Theorem 3.1).

For the first of these results we assume that X is a k-space [6], [7], [10], [12], [13], i.e. a space in which a set is closed if its intersection with every closed compact set is closed. The limitation to k-spaces is not a serious restriction. Every locally compact space is a k-space; so is every metrisable space. Although there do exist [6] completely regular spaces which are not k-spaces, such spaces do not seem to be important. Indeed Steenrod [10] has made a strong case for formulating topology entirely within the category of Hausdorff k-spaces.

At the same time as making the change from $C_0(X)$ to $B_0(X)$ it is natural to define the strict topology, in the first instance, on $B(X)$. In this form, our strict topology is a special case of generalizations introduced independently by Busby [2] and Sentilles and Taylor [9] in the context of Banach algebras.

Note added in proof. Recently, and independently, Gulick [14] and Sentilles [15] have also discussed the strict topology for $C(X)$, for completely regular X.

2. **Topologies on $B(X)$.** Let X be a set. Corresponding to each function ψ in $B(X)$ we define a seminorm p_ψ on $B(X)$ by writing, for every f in $B(X)$, $p_\psi(f) = \|\psi f\|$, where $\| \ \|$ denotes the sup norm.

Let $\Psi \subset B(X)$ be any subset. By the *Ψ-topology* on $B(X)$ we mean the topology determined by the set of seminorms $\{p_\psi : \psi \in \Psi\}$. A basis of open neighbourhoods for the Ψ-topology is $\{U_\psi : \psi \in \Psi\}$, where $U_\psi = \{f \in B(X) : p_\psi(f) < 1\}$.

Let $\Psi \subset B(X)$, $\Psi' \subset B(X)$, $\psi \in B(X)$, $\psi' \in B(X)$. If, for some constant λ, $\lambda|\psi'| \geqq |\psi|$ we say *ψ' dominates ψ*; if ψ' dominates every element of Ψ we say ψ' dominates Ψ. The proof of the following lemma is easy.

LEMMA 2.1. *If every element of Ψ is dominated by some element of Ψ' then the Ψ'-topology on $B(X)$ is finer than the Ψ-topology.*

LEMMA 2.2. *If ψ' dominates Ψ then $U_{\psi'}$ is a Ψ-bounded set (i.e. bounded in the Ψ-topology). Moreover, given any Ψ-bounded set $B \subset B(X)$ there is a ψ' dominating Ψ with $B \subset U_{\psi'}$.*

Proof. Given ψ in Ψ choose λ so that $\lambda|\psi'| \geqq |\psi|$. Then, for any f in $B(X)$, $p_\psi(f) \leqq \lambda p_{\psi'}(f)$, so that p_ψ is bounded on $U_{\psi'}$. Since this is true for every ψ, $U_{\psi'}$ is Ψ-bounded.

Now suppose $B \subset B(X)$ is Ψ-bounded. Then, for each ψ in Ψ we can choose λ_ψ such that $B \subset \lambda_\psi U_\psi$; clearly, we may also assume $\lambda_\psi \geqq \|\psi\|$. But then, for all ψ in Ψ,

$$f \in B \Rightarrow \|f\psi/\lambda_\psi\| < 1.$$

Thus $B \subset U_{\psi_0}$, where $\psi_0 = \sup\{|\psi|/\lambda_\psi : \psi \in \Psi\} \in B(X)$. Clearly, ψ_0 dominates each ψ in Ψ.

Corollary 2.3. *If every function which dominates* Ψ *dominates* Ψ' *then every* Ψ*-bounded set is* Ψ'*-bounded.*

Now let X be a topological space. We introduce four topologies on $B(X)$:

(a) σ, the uniform (or sup norm) topology, is the $B(X)$-topology or, equivalently (by Lemma 2.1), the $\{\mathbf{1}\}$-topology. (We denote by $\mathbf{1}$ the unit function on X.)

(b) β, our generalized strict topology, is the $B_0(X)$-topology.

(c) κ, the topology of compact convergence, is the Ψ_κ-topology, where $\Psi_\kappa=\{\psi\in B(X) : \psi \text{ has compact support}\}$.

(d) ρ, the topology of pointwise convergence, is the Ψ_ρ-topology, where $\Psi_\rho=\{\psi\in B(X) : \psi \text{ has finite support}\}$.

The following theorem gives the main properties of the strict topology β. The proofs are similar to those of Buck [1].

Theorem 2.4. *Let X be any topological space. Let the topologies σ, β, κ, ρ on $B(X)$ be defined as above. Then*

(i) $\sigma\supset\beta\supset\kappa\supset\rho$.

(ii) *If X is locally compact then β coincides with the strict topology as defined by Buck* [1].

(iii) *β and σ have the same bounded sets.*

(iv) *On any σ-bounded set B the topologies β and κ coincide.*

(v) *If X is a k-space then $C(X)$ is β-complete.*

Proof. (i) follows at once from Lemma 2.1.

(ii) Let K be locally compact. By Lemma 2.2, it is sufficient to show that each ψ in $B_0(X)$ is dominated by some ψ' in $C_0(X)$. We may clearly assume $\|\psi\|<1$. Choose compact sets K_n with $\varnothing=K_0\subset K_1\subset K_2\subset\cdots$ such that $|\psi(x)|<2^{-n}$ for $x\notin K_n$. Choose ψ_n in $C_0(X)$ with $\psi_n(x)=2^{-n}$ for $x\in K_n$ and $0\leqq\psi_n\leqq 2^{-n}\mathbf{1}$. Let $\psi'=\sum_{n=0}^{\infty}\psi_n$. Then $\psi'\in C_0(X)$ and ψ' dominates ψ.

(iii) Since $\sigma\supset\beta$ and since σ is the $\{\mathbf{1}\}$-topology on $B(X)$ it suffices, by Corollary 2.3, to show that if ψ_0 in $B(X)$ dominates $B_0(X)$ then ψ_0 dominates $\mathbf{1}$. Suppose, then, that ψ_0 does not dominate $\mathbf{1}$. Then there is a sequence (x_n) in X with $|\psi_0(x_n)|\to 0$. Let $\psi=\sum_{n=1}^{\infty}|\psi_0(x_n)|^{1/2}\chi(\{x_n\})$. Then $\psi\in B_0(X)$ but ψ_0 does not dominate ψ.

(iv) Choose M so that $\|f\|<M$ whenever $f\in B$. By (i), it suffices to show that the β-closure B_β of B contains the κ-closure B_κ of B. Suppose $g\in B_\kappa$. Given any ψ in $B_0(X)$ and any $\varepsilon>0$ choose a compact set $K\subset X$ with $|\psi(x)|<\varepsilon$ for $x\notin K$. Let $\psi'=\psi\chi(K)$. Then $\psi'\in\Psi_\kappa$ and, for every f in B, $p_\psi(f-g)=\|(f-g)\psi\|\leqq\|(f-g)\psi'\|+\|(f-g)(\psi-\psi')\|\leqq p_{\psi'}(f-g)+(M+\|g\|)\varepsilon$. Since $g\in B_\kappa$ and ε is arbitrary, this gives $\inf\{p_\psi(f-g) : f\in B\}=0$. Since this holds for all ψ in $B_0(X)$, $g\in B_\beta$.

(v) Let $\{f_\alpha\}$ be a β-Cauchy net in $C(X)$. By (i), $\{f_\alpha\}$ is κ-Cauchy, and hence [6], since X is a k-space, $f_\alpha\xrightarrow{\kappa}f$ where f is a continuous function on X. It remains to show that f is bounded and that $f_\alpha\xrightarrow{\beta}f$.

Now, for each ψ in $B_0(X)$, $\{\psi f_\alpha\}$ is a σ-Cauchy net in $B(X)$. Since $B(X)$ is σ-

complete, $\psi f_\alpha \xrightarrow{\sigma} g$ for some g in $B(X)$. But then $\psi f_\alpha \xrightarrow{\rho} g$ whereas also $\psi f_\alpha \xrightarrow{\rho} \psi f$ so that $g=\psi f$.

We have thus shown that, for each ψ in $B_0(X)$, $\psi f_\alpha \xrightarrow{\sigma} \psi f$ and $\psi f \in B(X)$. The first assertion implies that $f_\alpha \xrightarrow{\beta} f$ and, by Lemma 1.1, the second assertion means f is bounded.

3. **A Stone-Weierstrass theorem.** A Stone-Weierstrass theorem for $C(X)$ with the strict topology was established by Buck [1] subject to the condition that the algebra $\mathfrak{A}$ (see below) contains a function which vanishes nowhere. This condition was removed by Glicksberg [4] and later, in a simpler way, by Todd [11]. In all these cases the underlying space is, of course, locally compact. In this section we avoid the condition in a new way and do not impose any restriction on the space X.

THEOREM 3.1. *Let X be any topological space and $\mathfrak{A}$ be a β-closed subalgebra of $C(X)$ which separates points and contains, for each x in X, a function nonvanishing at x. Then $\mathfrak{A}=C(X)$.*

The proof uses

LEMMA 3.2. *Let $\varphi: \boldsymbol{R} \to \boldsymbol{R}$ be a continuous function with $\varphi(0)=0$. If $f \in \mathfrak{A}$ then $\varphi \circ f \in \mathfrak{A}$, where $\varphi \circ f$ is defined by $(\varphi \circ f)(x)=\varphi(f(x))$.*

Proof. Since $\mathfrak{A}$ is β-closed it is σ-closed. The lemma now follows from the Gelfand representation theorem for commutative C^*-algebras.

Proof of Theorem 3.1. Let $f \in C(X)$. We must show that, given any $\psi \in B_0(X)$, there exists f' in $\mathfrak{A}$ with $\|(f-f')\psi\|<1$. For it then follows that f is in the β-closure of $\mathfrak{A}$.

Choose M so that $M>\|f\|$ and $M>\|\psi\|$. Given $\varepsilon>0$ choose a compact set $K \subset X$ with $\psi(x)<\varepsilon$ for $x \notin K$. Clearly, $\mathfrak{A}$ separates points of K and does not vanish identically at any point of K. Hence, by the ordinary Stone-Weierstrass theorem, $\mathfrak{A}|K$ is σ-dense in $C(K)$. Choose $f'' \in \mathfrak{A}$ so that $\|(f-f'')|K\|<\varepsilon$.

Now assume $\varepsilon<M$ and let $\varphi: \boldsymbol{R} \to \boldsymbol{R}$ be defined by $\varphi(\lambda)=\lambda$ for $|\lambda| \leqq 2M$, $\varphi(\lambda) =2M$ for $\lambda>2M$, $\varphi(\lambda)=-2M$ for $\lambda<-2M$. Let $f'=\varphi \circ f''$. Then $\|f'\| \leqq 2M$ and, by Lemma 3.1, $f' \in \mathfrak{A}$. We now have $|[f(x)-f'(x)]\psi(x)|<\varepsilon M$ for $x \in K$, and $|[f(x)-f'(x)]\psi(x)|<3M\varepsilon$ for $x \notin K$, so that, by choosing $\varepsilon<1/3M$, we ensure $\|(f-f')\psi\|<1$. This completes the proof.

From Theorem 2.4(v) we have

COROLLARY 3.3. *If X is a k-space and $\mathfrak{A}$ is a subalgebra of $C(X)$ which separates points and vanishes identically nowhere then the β-closure of $\mathfrak{A}$ in $B(X)$ is $C(X)$.*

4. **The strict dual of $C(X)$.** For a locally compact space X, the dual space of $C(X)$ under the strict topology is the space of all finite regular signed Borel measures on X. We here extend this result to an arbitrary (not necessarily Hausdorff) completely regular space. Our measure-theoretic terminology is a simple generalization of that generally used [5] in the locally compact case.

DEFINITION 4.1. Let X be any topological space. By the *Borel sets* in X we mean the elements of the σ-algebra $\mathscr{B}(X)$ generated by the open sets. A *regular measure* on X is a (positive countably additive) measure on a σ-algebra $\mathscr{A} \subset \mathscr{B}(X)$ such that every A in $\mathscr{A}$ is

(a) *inner regular*, i.e.

$$\mu(A) = \sup\{\mu(K) : K \subset A, K \text{ closed compact}\},$$

and

(b) *outer regular*, i.e.

$$\mu(A) = \inf\{\mu(U) : A \subset U, U \text{ open}\}.$$

A signed measure μ is regular iff its total variation is regular.

If $\mu = \mu^+ - \mu^-$ is a Hahn decomposition of μ then μ is regular iff both μ^+ and μ^- are regular. Using this fact, properties of a regular signed measure can often be deduced from the case $\mu \geqq 0$.

LEMMA 4.2. *Let μ be a finite regular signed measure on a topological space X. For each function f in $C(X)$ let $L(f) = \int f\,d\mu$. Then L is a β-continuous linear functional on $C(X)$.*

Proof. It is sufficient to treat the case $\mu \geqq 0$. Since every element of $C(X)$ is a bounded Borel function and μ is finite, $L(f)$ is always defined and is linear. Assume for simplicity that $\mu(X) = 1$. Choose closed compact sets K_n, with $\varnothing = K_0 \subset K_1 \subset K_2 \subset \cdots$, such that $\mu(K_n) \geqq 1 - 2^{-2n}$.

Let $\psi = \sum 2^{-n}\chi(K_n)$. Then, for $x \in K_{n+1} - K_n$, $2^{-n-1} \leqq \psi(x) \leqq 2^{-n}$. The extended real-valued Borel measurable function $1/\psi$ is μ-integrable, indeed,

$$\int (1/\psi)\,d\mu = \sum_{n=0}^{\infty} [\mu(K_{n+1}) - \mu(K_n)]2^n \leqq \sum_{n=0}^{\infty} 2^{-n} = 2.$$

Now suppose $\varepsilon > 0$. Let $f \in C(X)$. Then, if $f \in U_{2\psi/\varepsilon}$, $\|2f\psi/\varepsilon\| < 1$ so that $|f| < \varepsilon/2\psi$ whence $|\int f\,d\mu| \leqq \int |f|\,d\mu \leqq \int (\varepsilon/2\psi)\,d\mu \leqq \varepsilon$. This proves the β-continuity of L.

In order to apply the Riesz representation theorem we now relate the regular Borel measures on a completely regular Hausdorff space to those on its Stone-Čech compactification.

LEMMA 4.3. *Let X be a completely regular Hausdorff space. We denote by βX its Stone-Čech compactification. For any regular signed Borel measure ν on βX we say ν satisfies the condition* (1) *iff*

$$|\nu|(\beta X) = \sup\{|\nu|(K) : K \subset X, K \text{ compact}\} \tag{1}$$

or, equivalently, iff there is a σ-compact set $J \subset X$ such that $|\nu|(J) = |\nu|(\beta X)$. Then

(a) *For any finite regular signed Borel measure μ on X let μ' denote the set function defined for each A in $\mathscr{B}(\beta X)$ by $\mu'(A) = \mu(A \cap X)$. Then μ' is a finite regular signed Borel measure on βX satisfying the condition* (1).

(b) *Conversely, if ν is any finite regular signed Borel measure on βX satisfying the condition* (1) *then there is a unique finite regular signed Borel measure μ on X such that $\nu=\mu'$.*

Proof. (a) Assume first that $\mu \geqq 0$. It is clear that μ' agrees with μ on $\mathscr{B}(X)$ and satisfies condition (1) by the regularity of μ. It remains to establish the regularity of μ'.

Let $A' \in \mathscr{B}(\beta X)$ and $A=A' \cap X$. Then $A \in \mathscr{B}(X)$ and

$$\begin{aligned}\mu'(A') &= \mu(A) = \sup\{\mu(K) : K \subset A,\ K \text{ compact}\} \\ &\leqq \sup\{\mu'(K) : K \subset A',\ K \text{ compact}\} \leqq \mu'(A').\end{aligned}$$

Thus A' is inner regular. On the other hand, since A is outer regular, given $\varepsilon>0$ there is a set V, relatively open in X, with $A\subset V\subset X$ and $\mu(V)<\mu(A)+\varepsilon$. Let V' be an open set in βX with $V' \cap X=V$, so that $\mu'(V')=\mu(V)$. By condition (1) there is a compact set $K\subset X$ with $\mu(X)-\mu(K)<\varepsilon$. Putting $W'=\beta X-K$, W' is open in βX and $\mu'(W')<\varepsilon$. Let $U'=V' \cup W'$. Since $W'\supset\beta X-X$, $U'\supset A \cup (\beta X-X)\supset A'$. Also U' is open and $\mu'(U')\leqq\mu'(V')+\mu'(W')<\mu(A)+2\varepsilon=\mu'(A')+2\varepsilon$. Thus A' is outer regular. This completes the proof in the case $\mu\geqq 0$. The general case follows easily by using a Hahn decomposition for μ.

(b) Any regular Borel measure is determined by its values on closed compact sets. The uniqueness of μ thus follows from the fact that if K is any compact subset of X then $K\in\mathscr{B}(\beta X)$ so that $\mu(K)=\nu(K)$. We establish the existence of μ. By condition (1) there is a σ-compact set $J\subset X$ with $|\nu|(\beta X-J)=0$. For the rest of the proof we consider first the case $\nu\geqq 0$. Let $\hat{\nu}$ be the completion of ν. Then X, and hence every Borel set in X, is $\hat{\nu}$-measurable. Let μ be the restriction of $\hat{\nu}$ to $\mathscr{B}(X)$. Clearly, $\hat{\nu}$ is regular and it follows easily from this that μ is regular. Lastly, for each A' in $\mathscr{B}(\beta X)$, $\nu(A')=\hat{\nu}(X \cap A')=\mu(X \cap A')$, which shows that $\mu'=\nu$.

For the general case ($\nu\geqq 0$) it is sufficient to observe that if $\nu=\nu^+ -\nu^-$ is a Hahn decomposition of ν, so that $|\nu|=\nu^+ +\nu^-$, then $\nu^+(\beta X-J)=\nu^-(\beta X-J)=0$. The above argument can then be applied to ν^+ and ν^- to obtain the positive and negative parts of μ.

Corollary 4.4. *The mapping $\mu\mapsto\mu'$ is a bijection between the finite regular signed Borel measures on X and those finite regular signed Borel measures on βX that satisfy the condition* (1). *Moreover, for every f in $C(X)$, $\int f\,d\mu=\int f'\,d\mu'$, where $f'\in C(\beta X)$ is the unique continuous extension of f.*

Proof. Let $J\subset X$ be a σ-compact set with $|\mu|(X-J)=0$. Then $|\mu'|(\beta X-J)=0$ too, while on J both the functions f and f' and the measures μ and μ' coincide.

We can now prove a converse to Lemma 4.2:

Lemma 4.5. *Let X be a completely regular space* (*not necessarily Hausdorff*) *and let L be a β-continuous linear functional on $C(X)$. Then there is a unique regular signed Borel measure μ on X such that $L(f)=\int f\,d\mu$ for all f in $C(X)$.*

Proof. First assume that X is Hausdorff. Since L is β-continuous it is certainly σ-continuous. Now, the canonical isomorphism of $C(X)$ onto $C(\beta X)$, which assigns to each bounded continuous function f on X its unique continuous extension f' on βX, is an isometry for the sup norm. So, by the Riesz representation theorem, there is a unique regular signed Borel measure ν on βX such that $L(f)=\int_{\beta X} f'\,d\nu$ for every f in $C(X)$.

We claim that the measure ν satisfies condition (1) of Lemma 4.3. Indeed, suppose that this condition is not satisfied. Then there is an $\varepsilon>0$ such that $|\nu|(\beta X-K)>\varepsilon$ for every compact set $K\subset X$. Now, since L is β-continuous there is a function ψ in $B_0(X)$ such that $\int_{\beta X} f'\,d\nu=L(f)<1$ whenever $f\in C(X)$ and $\|f\psi\|\leqq 1$. Let $K\subset X$ be a compact set such that $|\psi(x)|<\varepsilon$ for $x\in X-K$. Then $M=\beta X-K$ is a locally compact space, and the restriction ν_M of ν to $\mathscr{B}(M)$ is a bounded signed regular Borel measure on M. However, by the Riesz representation theorem, the set of all such measures is the dual $C_0(M)^*$ of the space $C_0(M)$ of all continuous real-valued functions vanishing at infinity on M. Regarded as an element of $C_0(M)^*$, ν_M has the norm $|\nu_M|(M)=|\nu|(\beta X-K)>\varepsilon$. Hence, by the Hahn-Banach theorem, there is a function f_M in $C_0(M)$ with $\|f_M\|<1$ and $\int_M f_M\,d\nu_M>\varepsilon$. Let f' in $C(\beta X)$ be the extension of f_M obtained by setting $f'(x)=0$ for $x\in K$ and let $f=f'|X$. Then $\int_{\beta X} f'\,d\nu>\varepsilon$. On the other hand $\|f\psi\|<\varepsilon$ whence, by the choice of ψ, $|\int_{\beta X} f'\,d\nu|<\varepsilon$, which is a contradiction.

It now follows from Lemma 4.3 that there is a bounded signed regular Borel measure μ on X such that $\mu'=\nu$ and we then have $L(f)=\int_X f\,d\mu$ for every f in $C(X)$.

Now suppose X is completely regular but not Hausdorff. Introduce the quotient space $Y=X/\sim$, where $x\sim y$ means $f(x)=f(y)$ for every f in $C(X)$. Then [8, p. 155] Y is completely regular and Hausdorff and the canonical map of X onto Y establishes a bijection between $\mathscr{B}(X)$ and $\mathscr{B}(Y)$ under which open sets and closed compact sets are preserved. There is thus a natural one-to-one correspondence between the regular signed Borel measures on X and those on Y. Moreover, the natural isomorphism [3, p. 41] of $C(Y)$ onto $C(X)$ is a homeomorphism for the strict topology—this follows from the easily established fact that the closure of each compact set in X is compact. The validity of the lemma for the space X is thus an immediate consequence of its validity for Y.

From Lemmas 4.2 and 4.5 we obtain

THEOREM 4.6. *For any completely regular space X the dual of $C(X)$ under the strict topology is the space of all bounded signed regular Borel measures on X.*

REFERENCES

1. R. C. Buck, *Bounded continuous functions on a locally compact space*, Michigan Math. J. **5** (1958), 95–104. MR **21** #4350.

2. R. C. Busby, *Double centralizers and extensions of C^*-algebras*, Trans. Amer. Math. Soc. **132** (1968), 79–99. MR **37** #770.

3. L. Gillman and M. Jerison, *Rings of continuous functions*, University Series in Higher Math., Van Nostrand, Princeton, N. J., 1960. MR **22** #6994.

4. I. Glicksberg, *Bishop's generalized Stone-Weierstrass theorem for the strict topology*, Proc. Amer. Math. Soc. **14** (1963), 329–333. MR **26** #4165.

5. E. Hewitt and K. Stromberg, *Real and abstract analysis*, Springer-Verlag, New York, 1965. MR **32** #5826.

6. J. L. Kelley, *General topology*, Van Nostrand, Princeton, N. J., 1955. MR **16**, 1136.

7. J. L. Kelley, I. Namioka et al., *Linear topological spaces*, University Series in Higher Math., Van Nostrand, Princeton, N. J., 1963. MR **29** #3851.

8. W. J. Pervin, *Foundations of general topology*, Academic Press Textbooks in Math., Academic Press, New York, 1964. MR **29** #2759.

9. F. D. Sentilles and D. C. Taylor, *Factorization in Banach algebras and the general strict topology*, Trans. Amer. Math. Soc. **142** (1969), 141–152. MR **40** #703.

10. N. E. Steenrod, *A convenient category of topological spaces*, Michigan Math. J. **14** (1967), 133–152. MR **35** #970.

11. C. Todd, *Stone-Weierstrass theorems for the strict topology*, Proc. Amer. Math. Soc. **16** (1965), 654–659. MR **31** #3891.

12. E. Wattel, *The compactness operator in set theory and topology*, Mathematical Centre Tracts, 21, Mathematisch Centrum, Amsterdam, 1968. MR **39** #7551.

13. D. D. Weddington, *On k-spaces*, Proc. Amer. Math. Soc. **22** (1969), 635–638. MR **40** #2001.

14. D. Gulick, *The σ-compact-open topology and its relatives* (preprint).

15. F. D. Sentilles, *Bounded continuous functions on a completely regular space* (preprint).

QUEEN'S UNIVERSITY, KINGSTON, ONTARIO, CANADA